Dynamics of Spinning Bodies Containing Elastic Rods

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The linearized partial and ordinary differential equations of spinning bodies containing elastic rods are formulated: the rods are along the spin axis and perpendicular to it. Rigid symmetric rotors parallel to the spin axis are included. After transformation to the frequency domain, the impedance of the rods is derived from the partial differential equations and then used to describe the dynamics of the combined system in terms of six degrees of freedom of the central body. No truncation error is induced as in normal mode analysis. The system becomes unstable when the lowest natural frequency approaches zero. So the stability limit is independant of internal damping. For rods along the spin axis and for cable-type rods perpendicular to the spin axis closed-form solutions are available. It is shown that the system becomes less stable if the translational motion of the center mass is suppressed.

Nomenclature

$\mathbf{A}$	= 6 × 6 impedance matrix
$\mathbf{a}$	= $\{a_i\}$; 3 × 1 vector of absolute velocity of rod point in components of β
EI	= flexural stiffness of rod
$\mathbf{f}$	= $\{M_i,K_i\}$; 6 × 1 vector of external forces in components of β
F_k, G_k	= impedance functions
h	= height of radial rod above $\beta_1\beta_2$ plane
I_i	= principal moments of inertia of undeformed structure
l	= length of rod
M	= mass of complete structure
m	= tip mass of rod
r	= distance of point attachment rod from spin axis (radial rod) or from center of mass (axial rod)
R	= momentum of rotor due to relative rotation
s	= La Place operator
$\mathbf{T}$	= transformation matrix
$\mathbf{u}$	= $\{u, v, w\}$; 3 × 1 vector of dimensionless elastic displacement in components of β
$\mathbf{v}$	= $\{\Omega_i,c_i\}$; 6 × 1 velocity vector of β in components of β
x	= coordinate on rod axis ($0 \leq x \leq 1$)
Z_k	= integrals of rod deflection
α	= inertial reference frame
β	= body fixed reference frame
ε	= $EI/(\bar{\Omega}_3{}^2\mu l^4)$ dimensionless flexural stiffness
μ	= rod mass per unit length
ν	= $\omega + (-)\bar{\Omega}_3$
ξ	= x
ρ	= r/l
φ	= angle between radial rod and $\beta_1\beta_3$ plane
$\bar{\Omega}_3$	= average of Ω_3
$\tilde{\Omega}_3$	= $\Omega_3 - \bar{\Omega}_3$
ω	= $-js$
$(\dot{\ })$	= derivative with respect to time
$(')$	= derivative with respect to spatial variable

Introduction

IN mechanics special interest has always been paid to the dynamics of rigid spinning bodies. If the spinning body cannot be treated as rigid, problems arise, the investigation of which is far from complete. Such problems occur in connection with spin stabilized satellites containing elastic rods, serving as antennas.

This study is the result of investigations carried out primarily for the project HELIOS. However, the results are given in the form of a parameter study and thus are not restricted to any special configuration. Figure 1 gives one possible configuration from the variety of systems covered by this study.

Received September 23, 1970; revision received October 26, 1971. This paper is based on a dissertation with K. Klöppel and R. Schardt, Technische Hochschule Darmstadt.

Index category: Spacecraft Attitude Dynamics and Control.

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The investigations reported in the literature are restricted to special structures. In Ref. 1 a symmetric spinning body containing two equal elastic rods placed in opposite directions along the spin axis, is investigated. In Ref. 2 attention is focused on a spinning body containing a symmetric rotor and a mass-spring-dashpot system. Both investigations indicate that elasticity can have a certain destabilizing effect. The purpose of this paper is to achieve generality by treating the effect of elasticity of various parts from the same point of view. It turns out that the stability limit can be given in an equally simple manner for spinning bodies with elastic rods as for rigid spinning bodies. This is true even for structures with an unrestricted number of radial rods, axial rods, dampers and rotors.

The following assumptions have been made: a rigid central body; straight rods extending in the direction of the centrifugal field or spin axis; arbitrary number and position of rods; parameters of rods constant over the entire length; rods may have tip-masses; no torsion and no elongation of rods; damping in rods linear but of arbitrary magnitude; rigid symmetric frictionless rotors; rotor axis parallel to spin axis but not necessarily coincident with it; small nutation cone (in order to linearize the problem); central body free to translate as well as rotate; and no gravitational forces, thermal effects, etc.

The dynamics of systems of the particular type under consideration here are described by sets of partial and ordinary differential equations. Six ordinary differential equations describe the rigid body motion while two partial differential equations per rod describe the rod deflections.

Usually the partial differential equations are converted into ordinary differential equations via normal mode expansion or spatial discretization.[1] This common practice has the disadvantage of generating a large number of equations and hence considerable computation. In addition, limiting the number of modes tends to make the mathematical model more stable than the actual system is. The error is small if the actual rod deflections can be closely approximated by the modal expansion which sometimes requires a large number of modes.

To overcome these difficulties, a method is chosen which is well known in principle, but is not commonly applied to the solution of the problem under consideration. In particular, an elastic rod is thought of as a black box. The analyst is

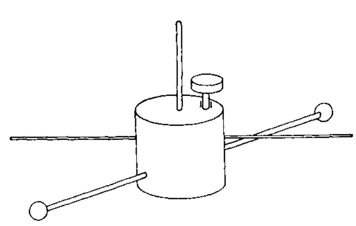

Fig. 1 Rigid body with flexible rods and rotor.

interested only in the input and the output of the box; what happens inside the box is irrelevant and unknown. The input of the box is represented by the excitation of the rod in terms of rigid body motions. The output is the reaction of the rod to the central body in terms of forces and moments exerted on the rigid central body. The deflections of the rod, though causing the forces and moments, remain inside the box. After transformation to the frequency domain, the relation between the input and output is given by a 6×6 impedance matrix which is generated by a computer subroutine, representing the inside of the black box.

This method offers the following advantages: No matter what the number and position of the rods, the partial differential equations must be set up and programmed for solution for one radial and one axial rod only. An entire variety of rods can be treated in the subroutine by varying the parameters.

Using the 6×6 impedance matrix, the dynamics of the coupled system (rigid body and elastic rods) can be formulated in terms of the six degrees of freedom of the rigid central body only. Any number of rods can be accounted for by repeatedly calling the subroutine. The method described is commonly applied to electrical and static mechanical problems and holds for linearized thermal problems as well.

The equations are formulated in a body-fixed frame of reference β, which is attached to the central rigid body. This frame of reference represents the principal axes of the undeformed system. Of course, the principal axes of the deformed body differ from the system β by rotation and translation. However, the principal axes of the deformed system are not only not needed, but the formualtion and solution of the problem become even simpler in terms of the coordinate system β.

The Lagrangian method is used to set up the equations. They contain terms due to the rigid central body as well as terms from the elastic rods. Among the latter, two different types of terms can be distinguished: those containing and those not containing elastic deformations. If the terms containing no elastic deformations are combined with the terms of the rigid central body, the system can be regarded as a rigid central body with rigid rods. Therefore, the problem is formulated in terms of a rigid body including rigid antennas and in purely elastic terms, which disappear in a rigid system.

The equations are linearized by assuming small nutation angle. Moreover the magnitude of the momentum vector is assumed constant. As a result, the spin rate consists of a constant and a harmonic component, the harmonic "ripple" being small compared with the constant "average" spin rate. Linearization is performed on all equations by dropping terms of order higher than one.

One of the most interesting questions is the stability of the system. The theory of stability by Liapunov starts with the nonlinear equations and calls the theory of the linearized variational system "theory in the first approximation." If the linearized system has only negative real parts in the characteristic exponents, then the nonlinear system is stable, and if at least one real part is positive, then the nonlinear system is unstable. If the real parts are all nonpositive, with at least one part being zero, a so-called critical case is encountered and the linearized system does not yield a conclusive stability statement. In the critical case stability is dictated by terms of higher than first order.

In the system encountered here, all real parts are, in general, zero. But this does not mean that terms of higher order will decide on stability, as disappearance of the real parts is only a consequence of having neglected damping. If damping is included in the analysis, all real parts will generally become different from zero and the linearized equations will yield the exact stability conditions of the non-linear system for small perturbations. The influence of internal damping on spinning bodies, however, was investigated and can be summarized as follows: Nutation of a spinning body makes the rotation vector form a cone in the spinning body. If this motion occurs in the direction of spin, damping has a stabilizing effect and vice versa. The common limit of both cases is the special case where the spin vector is fixed in the spinning body. This implies that the nutational frequency is zero, the motion is stationary and internal damping has no effect at all. These results are presented in a slightly different way in Ref. 3.

When investigating stability it is useful to recall that the characteristic polynomial of undamped systems has only terms of even powers. Thus, every eigenvalue multiplied by -1 is also an eigenvalue. Although the distribution of poles in the s-plane is always symmetric to the real axis it is also symmetric to the imaginary axis for undamped problems. From this it follows that the system is unstable if not all eigenvalues (= poles of the transfer function) lie on the imaginary axis of the s plane.

Varying the system parameters in order to find the conditions under which poles leave the imaginary axis shows that the poles can leave the imaginary axis only through the origin of the s plane. Although this fact was observed earlier, a formal proof is unknown to the author. On the other hand, poles can also go through the origin without leaving the imaginary axis. Whether or not the poles leave the imaginary axis when going through the origin depends solely on the question whether or not the transversal axes β_1 and β_2 are equivalent. The poles of symmetric systems (with equivalent axes β_1 and β_2) do not leave the imaginary axis; the poles of nonsymmetric systems do leave the imaginary axis.

If the poles do not leave the imaginary axis when going through the origin, the system will undergo nutational motion even after that event, but the rotation vector then will move in the opposite direction (as against the body-fixed system β) and the effect of damping will be contrary to that before the event.

These facts are of considerable practical interest. It can be inferred from them that the necessary and sufficient condition for the stability limit for an elastic spinning body with internal damping is given by setting the nutational frequency in the body-fixed reference frame equal to zero. Furthermore, they assure that this limit is independent of internal damping. After the nutational frequency has exceeded zero, symmetric systems will become unstable due to internal damping, whereas nonsymmetric structures become unstable regardless of internal damping.

The equations of motion are given in matrix form

$$\mathbf{A} \cdot \mathbf{v} = \mathbf{f} \tag{1}$$

The impedance maxtrix $\mathbf{A}$ includes the effects of the rigid body, radial rods, axial rods and rotors.

$$\mathbf{A} = \mathbf{A}_{rig} + \mathbf{A}_{rad} + \mathbf{A}_{axi} + \mathbf{A}_{rot} \tag{2}$$

The main task of the paper will be to set up these impedance matrices.

Differential Equations and Boundary Conditions

It will prove advantageous to choose a set of body axes coinciding with the principal axes of the undeformed structure and fixed with respect to the rigid central body. As this system is not inertial, derivatives with respect to time must be handled carefully.

The equations are set up for a rod parallel to the axis β_1. By a transformation to be described later, rods that differ from the case described here can be included. To give a concise presentation of the method, tip masses of the boom are omitted. However, there is no difficulty in including tip masses and the resuis are given including them.

The equations are set up using variational principles. To define the state of motion, 8 variables are necessary, 6 of which are discrete (3 translations and 3 rotations of the rigid central body) while 2 are continuous (deformations of the rod). The potential energy V and kinetic energy T of the system can be set up as functions of these 8 variables. The Lagrangian has the form

$$L = T - V \tag{3}$$

Hamilton's principle

$$\delta \int_{t_1}^{t_2} L dt = 0 \tag{4}$$

yields the Langrangian equations of the system. Because of the nature of the variables, 6 differential equations are ordinary and 2 are partial. The velocity **a** of point A' relative to the inertial system α, expressed in components of the body-fixed system β may be immediately derived from Fig. 2 in the matrix form

$$\mathbf{a} = \begin{bmatrix} \Omega_2(h + wl) - \Omega_3 vl & -\dot{u}l + c_1 \\ -\Omega_1(h + wl) + \Omega_3(r + (x - u)l) + \dot{v}l + c_2 \\ \Omega_1 vl & -\Omega_2(r + (x - u)l) + \dot{w}l + c_3 \end{bmatrix} \tag{5}$$

$$u(x) = \tfrac{1}{2}\int_0^x (v'^2 + w'^2) d\xi$$

Effecting the variation with respect to the two partial variables, certain mathematical difficulties will arise from the fact that the axial displacement u of the rod is an integral function of the lateral displacements v and w. Therefore, the partial differential equations are set up in a different way. This task is performed now, before proceeding with the formulation of the Lagrangian function.

The spin velocity is composed of a constant average portion $\bar{\Omega}_3$ and a varying portion $\tilde{\Omega}_3$

$$\Omega_3 = \bar{\Omega}_3 + \tilde{\Omega}_3 \tag{6}$$

where $\tilde{\Omega}_3$ is of the same order of magnitude as Ω_1 and Ω_2. Now the inertial acceleration of point A', expressed in terms of the body-fixed reference frame β, can be given.

$$(d/dt)\mathbf{a} = \begin{bmatrix} -\bar{\Omega}_3{}^2(r + xl) \\ \dot{c}_2 - \dot{\Omega}_1 h + \ddot{v}l + \bar{\Omega}_3(c_1 + \Omega_2 h - \bar{\Omega}_3 vl) + \\ (d/dt)(\tilde{\Omega}_3)(r + xl) \\ \dot{c}_3 - \dot{\Omega}_2(r + xl) + \ddot{w}l + \Omega_1\bar{\Omega}_3(r + xl) \end{bmatrix} \tag{7}$$

Only terms of the lowest order in every direction respectively are considered, as only they will form linear terms in the equations.

The differential equation for a rod with axial and lateral load is well known

$$(a_{EI}v'')'' + (a_N v')' + a_M\ddot{v} = a_L \tag{8}$$

where a_{EI}, a_N, a_M, and a_L represent the bending stiffness, normal force (pressure positive), mass distribution, and lateral loads, respectively.

In the problem encountered here, a_N can be given in terms of the first component of vector (7), while a_L results from the second and third component of vector (7). Starting with Eq. (8) the equations of the rod perpendicular to the spin axis are derived with the help of Eq. (7). The equations are

$$\varepsilon v'''' - [\rho(1 - x) + 0.5(1 - x^2)]v'' + (\rho + x)v' - v + \ddot{v}/\bar{\Omega}_3{}^2 = - [\bar{\Omega}_3(c_1 + \Omega_2 h) + \dot{c}_2 - \dot{\Omega}_1 h + (d/dt)(\tilde{\Omega}_3)(\rho + x)l]/(\bar{\Omega}_3{}^2 l) \tag{9a}$$

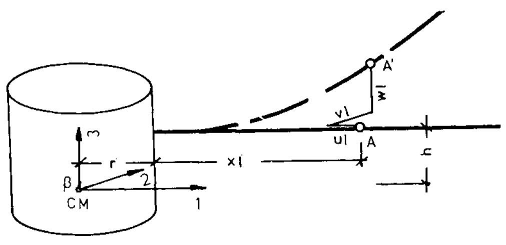

Fig. 2 Radial rod deflected.

$$\varepsilon w'''' - [\rho(1 - x) + 0.5(1 - x^2)]w'' + (\rho + x)w' + \ddot{w}/\bar{\Omega}_3{}^2 = - [l(\rho + x)(\bar{\Omega}_3\Omega_1 - \dot{\Omega}_2) + \dot{c}_3]/(\bar{\Omega}_3{}^2 l) \tag{9b}$$

The equations of the deformation of the rod in the equatorial plane v and in the meridian plane w differ by one term, as centrifugal acceleration is a radial field in the equatorial plane and a parallel field in the meridian plane.

Using variation principles, the Lagrangian equations can be derived from Eq. (4). Assuming ordinary variables, they are

$$(d/dt)(\partial L/\partial \dot{q}_i) - \partial L/\partial q_i = Q_i \tag{10}$$

In our case it is more convenient to use the Lagrangian equations for "quasi-coordinates."[4]

$$(d/dt)(\partial L/\partial\Omega_1) - \Omega_3(\partial L/\partial\Omega_2) + \Omega_2(\partial L/\partial\Omega_3) = M_1 \tag{11a}$$

$$(d/dt)(\partial L/\partial\Omega_2) - \Omega_1(\partial L/\partial\Omega_3) + \Omega_3(\partial L/\partial\Omega_1) = M_2 \tag{11b}$$

$$(d/dt)(\partial L/\partial\Omega_3) - \Omega_2(\partial L/\partial\Omega_1) + \Omega_1(\partial L/\partial\Omega_2) = M_3 \tag{11c}$$

$$(d/dt)(\partial L/\partial c_1) - \Omega_3(\partial L/\partial c_2) + \Omega_2(\partial L/\partial c_3) = K_1 \tag{11d}$$

$$(d/dt)(\partial L/\partial c_2) - \Omega_1(\partial L/\partial c_3) + \Omega_3(\partial L/\partial c_1) = K_2 \tag{11e}$$

$$(d/dt)(\partial L/\partial c_3) - \Omega_2(\partial L/\partial c_1) + \Omega_1(\partial L/\partial c_2) = K_3 \tag{11f}$$

The reference frame does not coincide with the principal axes of the deformed body, but this does not affect Eq. (9) provided the Lagrangian function is formulated correctly. With the help of Eq. (5) the kinetic energy T of the rod can be formulated.

$$T = 0.5\mu l \int_0^1 \mathbf{a}^2\, dx$$

The potential energy results from bending only, and can be written as

$$V = \tfrac{1}{2}EI\int_0^1 (v''^2 + w''^2)dx/l \tag{12}$$

According to Eq. (11), we obtain equations

$$-\mu l^2 h\int_0^1 \ddot{v}dx + \mu l^2 h\bar{\Omega}_3{}^2\int_0^1 v dx = M_1 \tag{13a}$$

$$-\mu l^2\int_0^1 \ddot{w}(r + xl)dx - 2\mu l^2\bar{\Omega}_3 h\int_0^1 \dot{v}dx - \mu l^2\bar{\Omega}_3{}^2\int_0^1 w(r + xl)dx = M_2 \tag{13b}$$

$$\mu l^2\int_0^1 (r + xl)\ddot{v}dx = M_3 \tag{13c}$$

$$-2\mu l^2\bar{\Omega}_3\int_0^1 \dot{v}dx = K_1 \tag{13d}$$

$$\mu l^2\int_0^1 \ddot{v}dx - \mu l^2\bar{\Omega}_3{}^2\int_0^1 v dx = K_2 \tag{13e}$$

$$\mu l^2\int_0^1 \ddot{w}dx = K_3 \tag{13f}$$

In Eq. (13) only terms that contain either v or w are included. All other terms represent the influence of the rigid rod. They are taken into consideration by adding the rigid rods to the rigid central body.

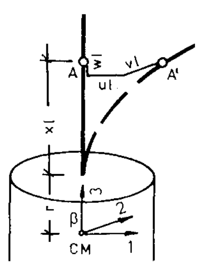

Fig. 3 Axial rod deflected.

The equations for the axial rod are derived in the same way. They have the form

$$\varepsilon u'''' + \ddot{u}/\bar{\Omega}_3{}^2 - u - 2\dot{v}/\bar{\Omega}_3 = -(\rho + x)(\Omega_1\bar{\Omega}_3 + \dot{\Omega}_2)/\bar{\Omega}_3{}^2 - \dot{c}_1/(\bar{\Omega}_3{}^2 l) + c_2/(\bar{\Omega}_3 l) \quad (14a)$$

$$\varepsilon v'''' + \ddot{v}/\bar{\Omega}_3{}^2 - v + 2\dot{u}/\bar{\Omega}_3 = -(\rho + x)(\Omega_2\bar{\Omega}_3 - \dot{\Omega}_1)/\bar{\Omega}_3{}^2 - \dot{c}_2/(\bar{\Omega}_3{}^2 l) - c_1/(\bar{\Omega}_3 l) \quad (14b)$$

$$\mu l^2 \int_0^1 (r + xl)(-\ddot{v} + \bar{\Omega}_3{}^2 v - 2\bar{\Omega}_3\dot{u})dx = M_1 \quad (14c)$$

$$\mu l^2 \int_0^1 (r + xl)(\ddot{u} - \bar{\Omega}_3{}^2 u - 2\bar{\Omega}_3\dot{v})dx = M_2 \quad (14d)$$

$$0 = M_3 \quad (14e)$$

$$\mu l^2 \int_0^1 (\ddot{u} - \bar{\Omega}_3{}^2 u - 2\bar{\Omega}_3\dot{v})dx = K_1 \quad (14f)$$

$$\mu l^2 \int_0^1 (\ddot{v} - \bar{\Omega}_3{}^2 v + 2\bar{\Omega}_3\dot{u})dx = K_2 \quad (14g)$$

$$0 = K_3 \quad (14h)$$

In Eq. (14), as in Eq. (13), only elastic terms are taken into consideration.

The boundary conditions are the same for the axial as well as for the radial rod.

$$v(0) = v'(0) = v''(1) = v'''(1) = 0 \quad (15)$$

Solution of the Partial Differential Equations

As the equations are linear with coefficients constant in time, derivatives with respect to time are removed by Laplace-transformation. It is useful to substitute

$$s = j\omega \quad (16)$$

Considering Eqs. (13), one notes that the deformations of the rod appear only in terms of their integrals over the rod length; the deformations as function of the rod length are not significant in this context. As the rigid body motions Ω_i and c_i appear as excitations in the partial differential Eqs. (9), the required integrals can be given as functions of the rigid body motions.

Equations (9) are linear but space dependent. They are integrated with respect to the spatial position by a computer subroutine.

The solutions of the partial differential Eqs. (9) are normalized by assigning the right sides the values 1 and $\rho + x$. The resulting deflections are marked by a (*) or a (**), respectively.

The following designations are introduced:

$$\int_0^1 v^* dx = Z_1; \quad \int_0^1 v^{**} dx = Z_2; \quad \int_0^1 v^*(\rho + x)dx = Z_3;$$
$$\int_0^1 v^{**}(\rho + x)dx = Z_4; \quad \int_0^1 w^* dx = Z_5; \quad \int_0^1 w^{**} dx = Z_6;$$
$$\int_0^1 w^*(\rho + x)dx = Z_7; \quad \int_0^1 w^{**}(\rho + x)dx = Z_8 \quad (17)$$

The values $Z_k = Z_k(\varepsilon, \rho, \omega)$ are computed by a program.

The matrix $\mathbf{A}_{\text{rad}}$ is obtained by substituting Eq. (17) into Eqs. (13).

$$\mathbf{A}_{\text{rad}} = \begin{bmatrix} \frac{\mu l h^2 j\omega(\omega^2 + \bar{\Omega}_3{}^2)Z_1}{\bar{\Omega}_3{}^2} & \frac{-\mu l h^2(\omega^2 + \bar{\Omega}_3{}^2)Z_1}{\bar{\Omega}_3} & \frac{-\mu l^2 h j\omega(\omega^2 + \bar{\Omega}_3{}^2)Z_2}{\bar{\Omega}_3{}^2} & \frac{-\mu l h(\omega^2 + \bar{\Omega}_3{}^2)Z_1}{\bar{\Omega}_3} & \frac{-\mu l h j\omega(\omega^2 + \bar{\Omega}_3{}^2)Z_1}{\bar{\Omega}_3{}^2} & 0 \\ \frac{2\mu l h^2\omega^2 Z_1}{\bar{\Omega}_3} \frac{-\mu l^3(\omega^2 - \bar{\Omega}_3{}^2)Z_8}{\bar{\Omega}_3} & \frac{2\mu l h^2 j\omega Z_1 + \mu l^3 j\omega(\omega^2 - \bar{\Omega}_3{}^2)Z_8}{\bar{\Omega}_3{}^2} & \frac{-2\mu l^2 h\omega^2 Z_2}{\bar{\Omega}_3} & 2\mu l h j\omega Z_1 & \frac{-2\mu l h\omega^2 Z_1}{\bar{\Omega}_3} & \frac{-\mu l^2 j\omega(\omega^2 - \bar{\Omega}_3{}^2)Z_7}{\bar{\Omega}_3{}^2} \\ \frac{-\mu l^2 h j\omega\omega^2 Z_3}{\bar{\Omega}_3{}^2} & \frac{\mu l^2 h\omega^2 Z_3}{\bar{\Omega}_3} & \frac{\mu l^3 j\omega\omega^2 Z_4}{\bar{\Omega}_3{}^2} & \frac{\mu l^2\omega^2 Z_3}{\bar{\Omega}_3} & \frac{\mu l^2 j\omega\omega^2 Z_3}{\bar{\Omega}_3{}^2} & 0 \\ \frac{2\mu l h\omega^2 Z_1}{\bar{\Omega}_3} & 2\mu l h j\omega Z_1 & \frac{-2\mu l^2\omega^2 Z_2}{\bar{\Omega}_3} & 2\mu l j\omega Z_1 & \frac{-2\mu l\omega^2 Z_1}{\bar{\Omega}_3} & 0 \\ \frac{-\mu l h j\omega(\omega^2 + \bar{\Omega}_3{}^2)Z_1}{\bar{\Omega}_3{}^2} & \frac{\mu l h(\omega^2 + \bar{\Omega}_3{}^2)Z_1}{\bar{\Omega}_3} & \frac{\mu l^2 j\omega(\omega^2 + \bar{\Omega}_3{}^2)Z_2}{\bar{\Omega}_3{}^2} & \frac{\mu l(\omega^2 + \bar{\Omega}_3{}^2)Z_1}{\bar{\Omega}_3} & \frac{\mu l j\omega(\omega^2 + \bar{\Omega}_3{}^2)Z_1}{\bar{\Omega}_3{}^2} & 0 \\ \frac{\mu l^2\omega^2 Z_6}{\bar{\Omega}_3} & \frac{-\mu l^2 j\omega\omega^2 Z_6}{\bar{\Omega}_3{}^2} & 0 & 0 & 0 & \frac{\mu l j\omega\omega^2 Z_5}{\bar{\Omega}_3{}^2} \end{bmatrix} \quad (18)$$

System (14) can be solved in a similar manner. A more direct solution, however, is based on the folowing idea. If the equations for the axial rod are formulated in an inertial coordinate system, as opposed to the body-fixed coordinate system used in Eq. (14), the well-known equation for a transversally vibrating rod

$$(a_{EI}v'')'' + a_M\ddot{v} = a_L \quad (19)$$

should appear. Equation (19) results from Eq. (8) by setting $a_N = 0$. This is so because the rotation of the rod has no influence on the motion since the moment of inertia of the rod about the spin axis is neglected. Now, closed-form solutions of Eq. (19) are known for different boundary conditions (e.g. Ref. 5). To employ these solutions for the problem encountered here, they must be brought into a suitable form. First, the excitation of the rod is transformed from the body-fixed to the inertial reference frame. The transformations are represented by rotation matrices relating the rotating and the nonrotating reference frames. Although they are not difficult in principle it would be tedious to demonstrate them in full detail. Thus, only two short remarks shall be given here. In Ref. 5, forces and moments are related to displacements. These formulas must be differentiated with respect to time, as the forces in relation to the appropriate velocities are needed here. In Ref. 5, the response of the complete rod is given, but here only that part of the response due to elastic deformations is needed. This gives rise to the terms λ in Eq. (20c).

The transformations result in the following impedance matrix for an axial rod:

$$\mathbf{A}_{\text{axi}} = \begin{bmatrix} (-G_1{}^m - G_1{}^p)j & -G_1{}^m + G_1{}^p & 0 & G_2{}^m - G_2{}^p & -G_2{}^m - G_2{}^p & 0 \\ G_1{}^m - G_1{}^p & (-G_1{}^m - G_1{}^p)j & 0 & (G_2{}^m + G_2{}^p)j & G_2{}^m - G_2{}^p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -G_2{}^m + G_2{}^p & (G_2{}^m + G_2{}^p)j & 0 & (-G_3{}^m - G_3{}^p)j & -G_3{}^m + G_3{}^p & 0 \\ (-G_2{}^m - G_2{}^p)j & -G_2{}^m + G_2{}^p & 0 & G_3{}^m - G_3{}^p & (-G_3{}^m - G_3{}^p)j & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (20a)$$

$$G_1 = EI(F_{15} - 2\rho F_{16} + \rho^2 F_{17})/(2\nu l)$$
$$G_2 = EI(F_{16} - \rho F_{17})/(2\nu l^2); \quad G_3 = EIF_{17}/(2\nu l^3) \tag{20b}$$

The terms F represent the solutions of Eq. (19) as given in Ref. 5.

$$\begin{aligned} F_{15} &= -\lambda(ch\lambda \sin\lambda - sh\lambda \cos\lambda)/(ch\lambda \cos\lambda + 1) + \lambda^4/3 \\ F_{16} &= \lambda^2 sh\lambda \sin\lambda/(ch\lambda \cos\lambda + 1) - \lambda^4/2 \\ F_{17} &= -\lambda^3(ch\lambda \sin\lambda + sh\lambda \cos\lambda)/(ch\lambda \cos\lambda + 1) + \lambda^4 \\ \lambda &= (\nu^2\mu l^4/EI)^{1/4}; \qquad \nu = \omega \pm \bar{\Omega}_3 \end{aligned} \tag{20c}$$

The superscript of G indicates which sign must be chosen in the equation for ν. Matrix (20a) is in accordance with a solution of Eq. (14) in the body-fixed reference frame.

Transformation of the Results to a Rod with Arbitrary Direction in the Equatorial Plane

Now a radial rod is considered which is derived from the former case by rotation with respect to axis β_3, by angle φ. This results in a transformation of the rod impedance matrix. The new matrix $\bar{\mathbf{A}}$ is given by

$$\bar{\mathbf{A}} = \mathbf{T}^{-1} \cdot \mathbf{A} \cdot \mathbf{T}$$

$$\mathbf{T} = \begin{bmatrix} c & -s & 0 & 0 & 0 & 0 \\ s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{21}$$

where c = cos and s = sin.

Equations of the Rigid Part

The rotational motion is described by the Eulerian equations

$$\begin{aligned} I_1\dot{\Omega}_1 + (I_3 - I_2)\Omega_3\Omega_2 &= M_1(t) \\ I_2\dot{\Omega}_2 + (I_1 - I_3)\Omega_1\Omega_3 &= M_2(t) \\ I_3\dot{\Omega}_3 + (I_2 - I_1)\Omega_2\Omega_1 &= M_3(t) \end{aligned} \tag{22}$$

The equations for the translational motion are derived with the help of the Lagrangian

$$\begin{aligned} (\dot{c}_1 - \Omega_3 c_2 + \Omega_2 c_3)M &= K_1(t) \\ (\dot{c}_2 - \Omega_1 c_3 + \Omega_3 c_1)M &= K_2(t) \\ (\dot{c}_3 - \Omega_2 c_1 + \Omega_1 c_2)M &= K_3(t) \end{aligned} \tag{23}$$

Assuming small nutation angle, Eq. (22) and Eq. (23) are linearized with the help of Eq. (6). After transformation to the frequency domain the impedance matrix can be given as follows:

$$\mathbf{A}_{rig} = \begin{bmatrix} j\omega I_1 & \bar{\Omega}_3(I_3 - I_2) & 0 & 0 & 0 & 0 \\ -\bar{\Omega}_3(I_3 - I_1) & j\omega I_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & j\omega I_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & j\omega M & -\bar{\Omega}_3 M & 0 \\ 0 & 0 & 0 & \bar{\Omega}_3 M & j\omega M & 0 \\ 0 & 0 & 0 & 0 & 0 & j\omega M \end{bmatrix} \tag{24}$$

The Equations of the Rotor

Rigid frictionless symmetric rotors are included in the analysis. The rotation axis is assumed to coincide with the rotor symmetry axis and to be parallel to β_3. Similar to the elastic rods it is advantageous to include the rotor when determining the rigid body parameters. Then the influence of the rotor can be given in terms of the momentum R due to the relative rotation only, leading to

$$\mathbf{A}_{rot} \left[\begin{array}{cc|c} 0 & R & 0 \\ -R & 0 & \\ \hline 0 & & 0 \end{array}\right] \tag{25}$$

Natural Frequencies and Stability

The conclusions of this chapter also hold for booms with tip masses. In the preceding chapters the impedance matrices of elastic rods and rigid body were set up separately. The impedance matrix of the combined system results from addition of these matrices. Now the matrix of the combined system will be used to obtain some results of practical interest.

To give the system response to an arbitrary continuous or discontinuous excitation it is necessary to find the transfer functions of the system. For a given system they differ only in the zeros and a constant factor. The poles are the same, as they represent the natural frequencies of the system.

These natural frequencies, however, are the zeros of the characteristic determinant calculated from the matrix **A**. The determinant also has poles, which represent the natural frequencies of the rods in the body-fixed reference frame when the body rotates about a inertially fixed axis. They are called "decoupled natural frequencies" of the rod. For $\varepsilon = 0$ and $\rho = 0$ the decoupled natural frequency of a radial rod becomes zero for vibrations in the equatorial plane and $\bar{\Omega}_3$ for vibrations in the meridian plane.

The decoupled natural freqencies of an axial rod are always arranged in pairs, with one natural frequency of a pair on either side of ω_n at a distance of $\bar{\Omega}_3$, ω_n being the decoupled natural frequency in a nonrotating system. For two equal radial rods opposite to one another, the motions c_3 and $\tilde{\Omega}_3$ will be decoupled from the remaining motion. The remaining terms are multiplied by 2. This results from application of Eq. (21) to Eq. (18). For two equal axial rods opposite to one another, the coupling between rotation and translation in Eq. (20a) will disappear. The remaining terms are multiplied by 2. If all radial rods are situated in the main equatorial plane ($h = 0$) in opposite but equal pairs, the coupling between rotation and translation will disappear in Eq. (18). In this case vibrations of the rods in the equatiorial plane will not be induced by nutation.

As pointed out earlier, the stability limit of the system is given by $\omega = 0$. A condition for those system parameters which make the system unstable is given by setting the determinant equal to zero for $\omega = 0$. This stability limit becomes especially simple if all rods extend in the directions of the 3 principal axes β_1, β_2 and β_3. Then an angular velocity (Ω_1 or Ω_2) is coupled only with the translational velocity perpendicular to itself (for $\omega = 0$). The coefficients of $\tilde{\Omega}_3$ and c_3 become zero for $\omega = 0$.

Due to the above decoupling, this special case permits a certain degree of understanding of the mechanics involved without simplifying the problem appreciably. Therefore, all the following investigations will be restricted to this case.

The following 2×2 matrix A_{stab} is set up for the pair $\Omega_1 c_2$ and gives the stability of the system with respect to rotation of the axis β_2. The matrix holds for the pair $\Omega_2 c_1$ correspondingly. Henceforth attention will be paid only to the rotation of axis β_2, bearing in mind that the corresponding requirements must be fulfilled for the stability of axis β_1

$$\mathbf{A}_{\text{stab}} = \bar{\Omega}_3 \left[\begin{array}{l:l} -(I_3 - I_1) - R/\bar{\Omega}_3 & -\sum [(\mu l + m)hZ_1]_j \\ +\sum [(\mu l + m)l^2 Z_8]_i & -\sum [(\mu l + m)lZ_{10}]_{k+} \\ +\sum [(\mu l + m)h^2 Z_1]_j & +\sum [(\mu l + m)lZ_{10}]_{k-} \\ +\sum [(\mu l + m)l^2 Z_9]_k & \\ \hdashline +\sum [(\mu l + m)hZ_1]_j & - \qquad M \\ +\sum [(\mu l + m)lZ_{10}]_{k+} & -\sum [(\mu l + m)Z_1]_j \\ -\sum [(\mu l + m)lZ_{10}]_{k-} & -\sum [(\mu l + m)Z_{11}]_k \end{array}\right] \tag{26a}$$

where

$$Z_9 = (-F_{15} + 2\rho F_{16} - \rho^2 F_{17})/\lambda^4$$
$$Z_{10} = (F_{16} - \rho F_{17})/\lambda^4; \quad Z_{11} = (-F_{17})/\lambda^4 \tag{26b}$$

The values of F and λ are given by (20c).

Moreover, i, j and k pertain to axes β_1, β_2 and β_3, respectivly, where $k+$ refers to $+\beta_3$ and $k-$ refers to $-\beta_3$.

The integrals Z_1, Z_8, and Z_9 are given in Figs. 5–7 for $m = 0$. If the bending stiffness of the boom is negligible (cablebooms), analytical expressions can be given for Z_1 and Z_8. Hence, for $\varepsilon = 0$ and $\omega = 0$ we obtain

$$Z_1 = 0.5[1 + m/(\mu l + m)]l/r$$
$$Z_8 = 1/3 + 0.5r/l + (2/3 + 0.5r/l)m/(\mu l + m) \tag{27}$$

The system is stable if the determinant associated with Eq. (26a) is positive. It is useful to restrict Eq. (26a) to some more special cases. If the origin of β is inertially fixed, the determinant reduces to the upper left square. The reduced determinant is positive if

$$I_3 - I_1 - \sum [(\mu l + m)l^2 Z_8]_i - \sum [(\mu l + m)h^2 Z_1]_j - \sum [(\mu l + m)l^2 Z_9]_k + R/\bar{\Omega}_3 \geq 0 \tag{28}$$

If the origin of β is free to move, a positive (i.e. stabilzing) term is added to the left side of Eq. (28). This becomes evident when forming the determinant of Eq. (26a).

Further insight into the problem can be gained if the parameters of the central body without booms are introduced. They shall be designated by $(\tilde{I}_1, \tilde{I}_2, \tilde{I}_3, \tilde{M})$.

By the addition of rods to the central body, the principal axes of the undeformed system will be altered. To keep the formulas simple, this change has been restricted to a pure translation in the direction of the spin axis β_3. Furthermore, only cable-type radial booms will be investigated. During motion the origin of β is to be inertially fixed. For this case the stability condition has the form:

$$\tilde{I}_3 - \tilde{I}_1 - \Delta I_1 + \sum \{(\mu l + m)[0.5 + r/l + 0.5m/(\mu l + m)]rl\}_i - \sum \{(\mu l + m)h^2[0.5 + 0.5m/(\mu l + m)]l/r\}_j + R/\bar{\Omega}_3 \geq 0 \tag{29a}$$

where

$$\Delta I_1 = \tilde{M}h^2 + \sum [(\mu l + m)h^2]_i + \sum [(\mu l + m)h^2]_j \tag{29b}$$

h is the height of the respective part above the main equatorial plane of the combined system. Of course, all formulas also hold for negative r, ΔI_1 is due to the shift of the mass center

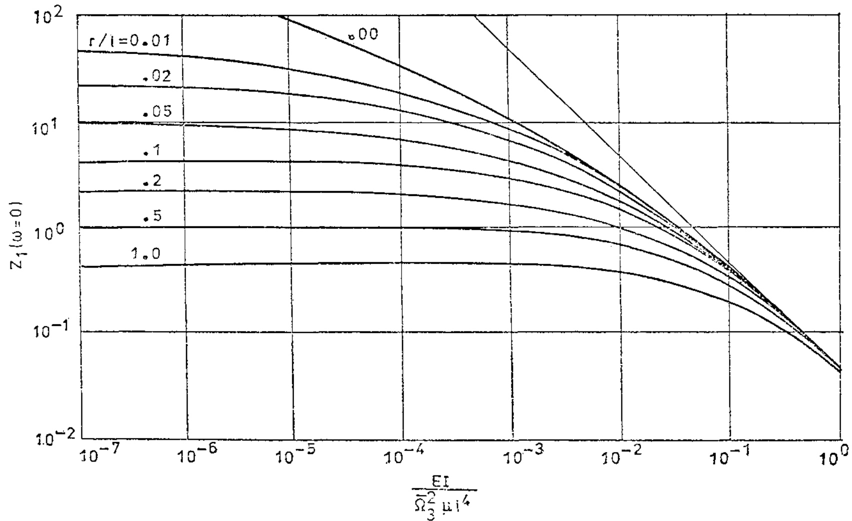

Fig. 5 Parameter plot of coefficients Z_1.

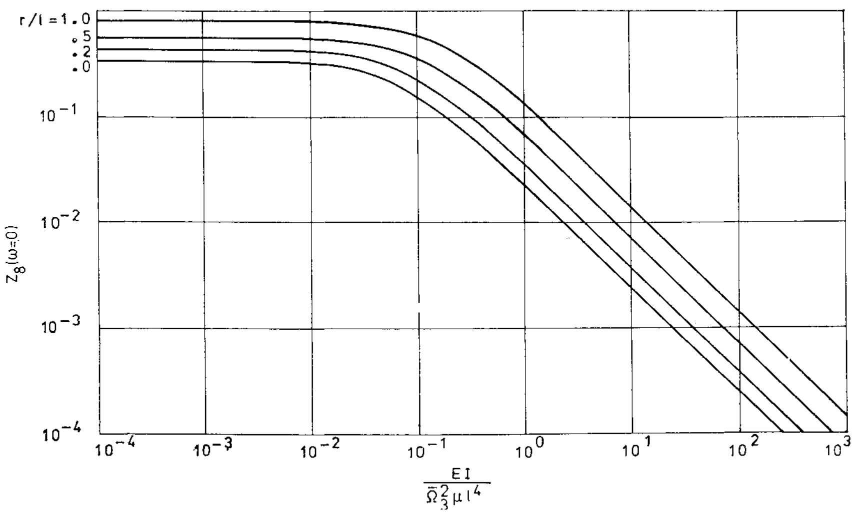

Fig. 6 Parameter plot of coefficients Z_8.

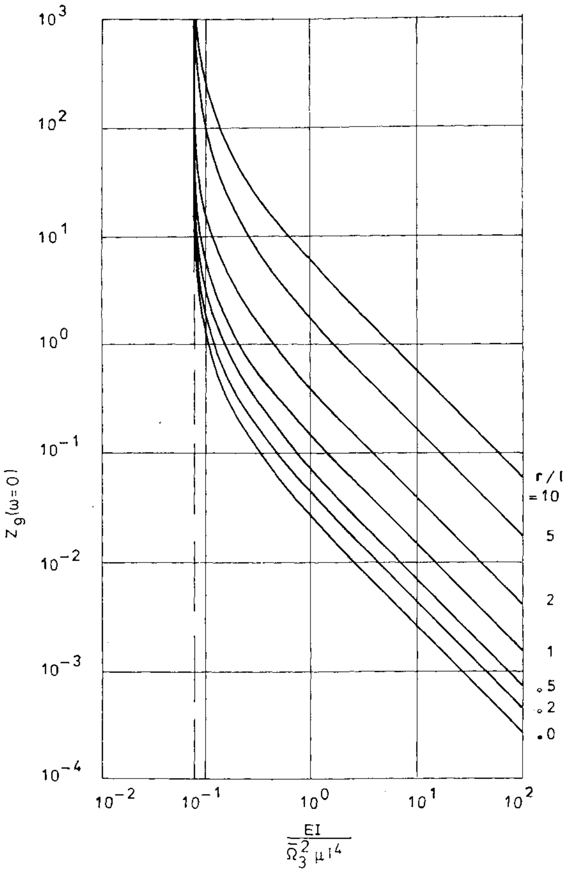

Fig. 7 Parameter plot of coefficients Z_9.

along the spin axis and can be generally neglected. The main terms of Eq. (29a) make clear that the addition of a cabletype boom to a rigid body will stabilize the system with respect to the axis perpendicular to the boom and will destabilize the system with respect to the axis parallel to the boom.

In Ref. 1, the stability of a spinning body with two symmetric axial rods is investigated. The system is described by two different models. In Model 1 the rods are substituted by masspoints m, suspended in the spin axis by springs at a distance a from the main equatorial plane. In Model 2 the rods are not subject to mechanical simplifications; the mobility of the rods is restricted, however, by the introduction of a finite number of normal modes. The stability of the models is investigated by evaluating the roots of the characteristic polynomials with the help of a computer program.

As pointed out earlier, the stability limit is given by $\omega = 0$ for problems of the type encountered here. Thus the system becomes unstable when the absolute term of the characteristic polynomial changes its sign. Therefore, the stability limit can be given in closed form. The formula for Model 1 is

$$\omega_n^2/\bar{\Omega}_3^2 \geq 1 + 2ma^2/(I_3 - I_1) \tag{30}$$

where ω_n is the natural frequency in a nonrotating system.

In Ref. 1 the following notation is introduced for Model 1:

$$I_1 = A(1 + R_1); \quad 2ma^2 = R_1 A; \quad I_3 = C$$

With these designations Eq. (30) reads as follows:

$$\omega_m^2/\Omega_3^2 \geq (C/A - 1)/(C/A - 1 - R_1) \tag{31}$$

Eq. (31) is given in Ref. 1 in the form of a parameter plot.

The stability limit of the continuous model is also given in Ref. 1. This plot holds for only one normal mode, although the inclusion of one more mode does not appreciably alter the

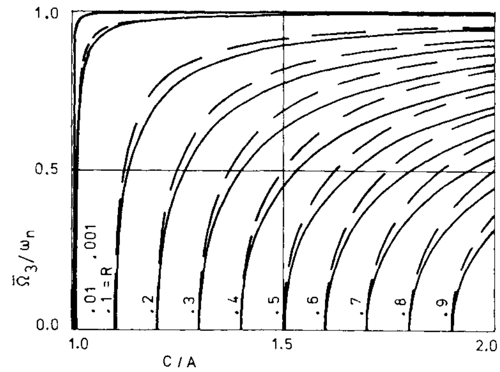

Fig. 4 Parameter plot of Eqs. (31) (continuous line) and Eqs. (34) (dotted line).

plot. The exact stability limit of the continuous model can be given with the help of Eq. (28)

$$I_3 - I_1 - 2\mu l^3 Z_9 \geq 0 \tag{32}$$

In Ref. 1, the following notation is introduced for Model 2:

$$I_1 = A(1 + R_2); \quad R_2 = 2\mu l^3(1/3 + \rho + \rho^2)/A$$
$$I_3 = C; \quad \bar{\Omega}_3^2/\omega_1^2 = 0.0807/\varepsilon \tag{33}$$

where ω_1 is the first natural frequency of the rod in a nonrotating system.

With this notation, Eq. (32) reads as follows:

$$C/A - 1 - R_2[1 + Z_9/(1/3 + \rho + \rho^2)] \geq 0$$
$$Z_9 = [-F_{15}(\lambda) + 2\rho F_{16}(\lambda) - \rho^2 F_{17}(\lambda)]/\lambda^4 \tag{34}$$
$$\lambda^4 = (\bar{\Omega}_3/\omega_1)^2/0.0807$$

The values of F_{15-17} are given by Eqs. (20c). Figure 4 is in good agreement with the parameter plot given in Ref. 1. This is due to the fact that the normal mode chosen in Ref. 1 differs only slightly from the actual rod deformation.

A ball-in-tube nutation damper may be regarded as a masspoint on a massless cable type boom. Consequently, the destabilizing effect of such a damper may be calculated with Eq. (29a). The result is in accordance with the stability limit given for instance in Ref. 6.

In Refs. 2 and 7, the stability of spinning bodies with symmetric rotors and dampers is investigated. Exact solutions are given as well as approximations for the quasi-rigid case with negligible damper mass. In both cases the results are in accordance with those given here.

Summary and Conclusions

The linearized partial and ordinary differential equations of spinning bodies containing elastic rods along and perpendicular to the spin axis are formulated. The impedance of the rods at the attachment points to the central body is derived from the partial differential equations, and is used to describe the dynamics of the combined system in terms of the six degrees of freedom of the central body. It turns out that the only stability limit of the system is given by $\omega = 0$ and thus it is independent of damping. The stability limit can be readily determined with the help of diagrams; for cable-type booms (bending stiffness negligible), closed-form solutions can be given.

With the method demonstrated herein, systems with an unlimited number of radial rods in different directions can be treated easily. No restrictions as to symmetry of system or deflections of rods are imposed on the system. The mobility of the rods is not restricted to a certain number of modes.

Special structures within the range and variety of systems covered by this investigation have been studied earlier. The results given in the literature are in good agreement with the results obtained by the method described here. Closed-form solutions are given for some stability diagrams. These were obtained in the literature with the help of computers.

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APRIL 1972 J. SPACECRAFT VOL. 9, NO. 4

Behavior of a Two Degree-of-Freedom Gyroscope in a Rotating Satellite

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The equations of motion of a two degree-of-freedom gyroscope mounted in an artificial satellite are used to formulate conditions under which the spin-axis of the rotor can remain nearly fixed in inertial space. As an illustrative application, an attitude control scheme for Earth-pointing satellites is devised.

Nomenclature

a	= distance from Q to T
$\mathbf{a}_1$	= unit vector parallel to QT
$\mathbf{a}_2$	$= \mathbf{a}_3 \times \mathbf{a}_1$
$\mathbf{a}_3$	= unit vector parallel to E_3
b	= distance from P to T
$\mathbf{b}_i (i = 1,2,3)$	= unit vectors parallel to principal axes of inertia of B for P
$B_i (i = 1,2,3)$	= principal moments of inertia of B for P
$\mathbf{c}_i (i = 1,2,3)$	= unit vectors parallel to principal axes of inertia of C for P
$C_i (i = 1,2,3)$	= principal moments of inertia of C for P
D_1	= moment of inertia of D about any line passing through P and perpendicular to $\mathbf{c}_2$
D_2	= moment of inertia of D about symmetry axis
$\mathbf{e}_i (i = 1,2,3)$	= unit vector parallel to E_i
R	= radius of circular orbit
α	= angle between $\mathbf{e}_1$ and $\mathbf{a}_1$
β	= angle between $\mathbf{a}_1$ and $\mathbf{b}_1$
γ	= angle between $\mathbf{b}_2$ and $\mathbf{c}_2$
σ	= angular speed of D relative to C
Ω	= angular speed of line $O - Q$ in an inertial reference frame

Introduction

TWO degree-of-freedom gyroscopes are employed extensively in inertial navigation and guidance systems. It has been shown that the spin-axis of such a gyroscope does not necessarily have a fixed orientation in inertial space when the gyroscope is mounted in a rotating vehicle.[1] Hence, it is important that vehicle motions be taken into account when dealing with such systems.

It is the purpose of this paper to show how one can cause the spin-axis of a two degree-of-freedom gyroscope that is mounted in a rotating satellite to remain nearly fixed in inertial space. The results obtained are used to devise an attitude control scheme for Earth-pointing satellites, and it is demonstrated that this scheme leads to significant improvements in roll and yaw control.

System Description

In Fig. 1, O designates a particle fixed in an inertial reference frame, and A is an artificial satellite whose mass center, Q, moves with constant angular speed in a circular orbit

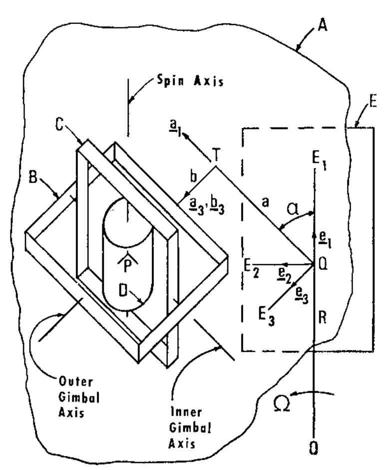

Fig. 1 Satellite and gyroscope.

Received July 26, 1971; revision received November 16, 1971.
Index category: Spacecraft Attitude Dynamics and Control.

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